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The twistor space of the conformal six sphere and vector bundles on quadrics

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Abstract

The twistor fibration $\tau : Q_+ \to S^6$ fibres the non-degenerate, six-dimensional complex quadric hypersurface Q_+ over the conformal six-sphere S^6 with fiber $P_3(\mathbb{C})$. In this paper we show that the map τ induces an isomorphism of one component of the space of linear $P_3(\mathbb{C})$'s lying on Q_+ which are not fibres onto the space of oriented conformal four-spheres of S^6; further, this map lifts to a map between the corresponding tautological bundles which fibre by fibre is the usual Penrose twistor fibration $P_3(\mathbb{C}) \to S^4$. It is also shown that a holomorphic vector bundle over a non-degenerate, complex quadric hypersurface of dimension greater than or equal to six is trivial if and only if its restriction to a linear $P_2(\mathbb{C})$ is trivial.

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1. Introduction

The Penrose transform establishes a correspondence between the conformal geometry of the standard four-sphere S^4 and the holomorphic geometry of projective lines on $\mathbf{P}_3(\mathbb{C})$. The complex manifold $\mathbf{P}_3(\mathbb{C})$ is obtained from S^4 as a fibre bundle with fibre $\mathbf{P}_1(\mathbb{C})$ by the twistor space construction and given $\mathbf{P}_3(\mathbb{C})$, there is also a way of recovering S^4 and its conformal structure (cf. [AHS] for example). The twistor space construction can be generalised to any even-dimensional, oriented manifold equipped with a conformal structure (cf. [BeO], [I], [O'BR] or [S]) and in this paper we will look at what happens when it is applied to the standard six-sphere S^6. The complex manifold that we obtain (cf. [I,W]) is the six-dimensional, non-degenerate quadric hypersurface Q_+ fibred over S^6 with fibre $\mathbf{P}_3(\mathbb{C})$. It is

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classical (cf. [GH] for example) that there are two families of linear $\mathbf{P}_3(\mathbb{C})$'s lying on Q_+, say Q and Q_-. The first main result of this paper, refining results of Inoue in [I], describes one of these families, which we will take as Q and which is the family containing the fibres, in terms of the conformal geometry of S^6.

Theorem 1.1.
(i) *If P is a $P_3(\mathbb{C})$ belonging to the family parametrised by Q, then P is either a fibre of the twistor fibration $\tau : Q_+ \to S^6$ or its image $\tau(P)$ is a conformal four sphere (cf. [I]); furthermore $\tau : P \to \tau(P)$ is isomorphic to the Penrose fibration for a suitable orientation of $\tau(P)$.*
(ii) *Over each conformal four sphere S in S^6, there are exactly two $P_3(\mathbb{C})$'s of the family Q. They induce opposite orientations on S and intesect each fibre of $\tau : Q_+ \to S^6$ over S in two disjoint $P_1(\mathbb{C})$'s. They are the images in Q_+ of the twistor lifts (cf. Definition 4.8) of the inclusion $S \hookrightarrow S^6$.*

Thus τ maps the space of $P_3(\mathbb{C})$'s in the family Q which are not fibres isomorphically onto the space of oriented conformal four-spheres in S^6, and lifts to a map between the corresponding tautological bundles which fibre by fibre is the usual Penrose twistor fibration of $P_3(\mathbb{C})$ over S^4.

To prove this theorem, we realise the conformal sphere S^6 as the projective isotropic cone of a real, eight-dimensional inner product space of Lorentz signature (7, 1). Spinor spaces of this Lorentz space then provide a convenient way of modeling the twistor fibration of S^6, and of parametrising the above families of linear $P_3(\mathbb{C})$'s on its twistor space.

As an application, we prove the following theorem and corollary.

Theorem 1.2. *If $\xi \to Q_+$ is a holomorphic vector bundle such that the restriction to every fibre of $\tau : Q_+ \to S^6$ is holomorphically trivial, then $\xi \to Q_+$ is holomorphically trivial.*

Corollary 1.3. *Let G be a non-degenerate, complex quadric hypersurface of dimension greater than or equal to six. If $\xi \to G$ is a holomorphic vector bundle such that ξ restricted to a linear $P_2(\mathbb{C})$ is holomorphically trivial, then $\xi \to G$ is holomorphically trivial.*

There is at least one aspect of the Penrose transform for four-manifolds which does not generalise to dimension six. For instance, the $P_3(\mathbb{C})$ realised as the Penrose transform of S^4 comes with a real structure, that is an antiholomorphic fibre-preserving involution σ which has no fixed points. This is important because it is using σ that one recovers S^4 and its conformal structure from $P_3(\mathbb{C})$ (cf. [AHS]). The twistor space of S^6 has no such involution. However, it is shown in [I] that the fibre component of the space of $P_3(\mathbb{C})$'s on the twistor space of S^6 does have a natural, antiholomorphic involution, and that one can then recover S^6 and its conformal geometry as the fixed point set. In Section 5 we give an intrinsic definition of this real structure using the notion of the twistor lift of a codimension two, immersed, oriented submanifold.

As a corollary of the observation that the twistor space of S^6 is Kähler and a theorem of Burns-de Bartolomeis [BdeB], we give a conformal proof of the following theorem of LeBrun [LeB].

Theorem 1.4. *There does not exist an integrable, almost complex structure on S^6 which is compatible with the standard conformal structure.*

2. Clifford algebras and spinors

In this section we will recall some basic facts about Clifford algebras and spinors, only proving those which are not completely standard. For more details, see Cartan's book [C].

If W is a complex vector space equipped with a non-degenerate, symmetric bilinear form g, the Clifford algebra $C(W,g)$ is defined as the quotient algebra $T(W)/\Im_g$, where $T(W)$ is the tensor algebra on V and $\Im_g$ is the two-sided ideal generated by elements of the form $w \otimes w + g(w,w)1$. The natural map $\Lambda(W) \to T(W) \to C(W,g)$ is a vector space isomorphism equivarient for the natural action of the orthogonal group $O(W,g)$ and if $e_1, e_2, \ldots, e_n$ is an orthogonal basis of W, the algebra $C(W,g)$ is generated by their images via this inclusion, subject to the relations.

$$e_i e_j + e_j e_i = -2g(e_i, e_j)1.$$

The Clifford algebra has the following universal property: every linear map $f : W \to A$ from W to an associative algebra with identity A such that $f(w)^2 = -g(w,w)1_A$ for all $w \in W$, extends to a unique algebra homomorphism $\tilde{f} : C(W,g) \to A$.

The natural $\mathbb{Z}_2$-grading of $T(W)$ into even and odd degree tensors induces a $\mathbb{Z}_2$-grading of the Clifford algebra $C(W,g) = C_+ \oplus C_-$. More generally, any automorphism (resp. anti-automorphism) of $T(W)$ which preserves the ideal $\Im_g$ induces an automorphism (resp. anti-automorphism) of $C(W,g)$. In particular, we will write $x \mapsto x^{\mathrm{t}}$ for the anti-automorphism induced by $w_1 \otimes w_2 \otimes \cdots \otimes w_k \mapsto w_k \otimes w_{k-1} \otimes \cdots \otimes w_1$, and if $\sigma : W \to W$ is a real form of (W,g), we write $x \mapsto \bar{x}$ for the induced conjugate linear automorphism of $C(W,g)$ which extends σ.

When dim $W = 2m$ is even, one can show that $C(W,g)$ is a full matrix algebra and if we choose a complex vector space S of dimension 2^m and an algebra isomorphism $C(W,g) \cong \mathrm{End}(S)$, we call S a space of spinors for (W,g). The various natural automorphisms of $C(W,g)$ are realised by geometric structures on S: there is a $\mathbb{Z}_2$-grading $S = S_+ \oplus S_-$ such that $C_+ \cdot S_\pm \subseteq S_\pm$ and $C_- \cdot S_\pm \subseteq S_\mp$; there is a (unique upto a constant factor) non-degenerate, bilinear form $B_S : S \times S \to \mathbb{C}$ such that $B_S(x \cdot \psi, \phi) = B_S(\psi, x^{\mathrm{t}} \cdot \phi)$ and then $B_S(\psi,\phi) = (-1)^{m(m-1)/2} B_S(\phi,\psi)$; there is a (unique upto phase factor $\mathrm{e}^{\mathrm{i}\theta}$) conjugate linear $j : S \to S$ such that $j(x \cdot \psi) = \bar{x} \cdot j(\psi)$ and $j^2 = \pm \mathrm{Id}_S$ (the sign will depend on the signature of σ).

In particular, if dim $W = 6$ and σ is of Lorentz signature (5, 1), we find that $j^2 = -\mathrm{Id}$ and so j induces a quaternionic structure on each of the four-dimensional complex vector space S_+ and S_-. The classical Penrose twistor fibration (cf. [AHS]) $T : \mathbf{P}_3(\mathbb{C}) \to S^4$ can be recovered as follows: take a non-zero positive spinor $\psi \in S_+$ and let $[\psi] \in \mathbf{P}(S_+)$ be the corresponding complex line; then $\{w \in W : w \cdot \psi = \sigma(w) \cdot \psi = 0\}$ is a real, isotropic (since $w^2 \cdot \psi = -g(w, w)\psi = 0$) line in W and so defines a point of the real, projective isotropic cone of (W, g, σ), which is well known to be isomorphic to the standard conformal four-sphere S^4. The map which sends $[\psi]$ to this point is the Penrose twistor fibration. (The analogous process starting from the other projective spinor space $\mathbf{P}(S_-)$ gives the Penrose fibration of the same four-sphere but with respect to the opposite orientation.) This point of view will be explained more fully in Section 3.

Let us now examine the situation in eight dimensions in more detail. To fix the notation, let V be an eight-dimensional, complex vector space and let B be a non-degenerate, symmetric bilinear form on V. Let C be the Clifford algebra of (V, B) and if we choose a spinor space Σ, then

$$\Sigma = \Sigma_+ \oplus \Sigma_-$$

will be the decomposition of the 16-dimensional spinor space Σ as the sum of the two eight-dimensional semi-spinor spaces. According to the general discussion above, we can choose a non-degenerate, symmetric bilinear form B_Σ on Σ such that

$$B_\Sigma(x \cdot \psi, \phi) = B_\Sigma(\psi, x^{\mathrm{t}} \cdot \phi), \quad \text{where } x \in C;\ \psi, \phi \in \Sigma.$$

It can be shown that $B_\Sigma(\Sigma_+, \Sigma_-) = 0$ (see [C]) so that each of the spaces $\Sigma_\pm$ has its own symmetric bilinear form $B_\pm = B|\Sigma_\pm$. Notice that each of the spaces $\Sigma_\pm$ is an eight-dimensional complex vector space equipped with a symmetric, non-degenerate bilinear form which is exactly the kind of object which we started with. This is a manifestation of the symmetry which exists in dimension eight between the three types of objects: vectors (V), positive semi-spinors (Σ_+) and negative semi-spinors (Σ_-). This symmetry is often referred to as the 'principle of triality' (cf. [C]).

Notation 2.1. We will denote by Q the space of isotropic lines in V. This is a smooth, six-dimensional quadric hypersurface in the projective space $\mathbf{P}(V)$:

$$Q = \{L \in \mathbf{P}(V) : B|L = 0\}.$$

The spaces $Q_\pm$ are defined similarly—they are smooth quadric hypersurfaces in $\mathbf{P}(Q_\pm)$, isomorphic to Q as complex manifolds. If $x \in V$ is a non-zero vector, then $[x] \in \mathbf{P}(V)$ will be the corresponding line.

Lying on Q there are two families of $\mathbf{P}_3(\mathbb{C})$'s given by the images in $\mathbf{P}(V)$ of maximal isotropic subspaces of (V, B). By the theory of spinors (cf. [C]), the space of maximal isotropic subspaces of V is parametrised by the disjoint union $Q_+ \cup Q_-$: if $[\psi] \in Q_\pm$, then $\mathrm{Ann}(\psi) = \{x \in V : x \cdot \psi = 0\}$ is maximal isotropic and any maximal isotropic subspace of V is uniquely obtained in this way.

The quadric Q has a natural complex conformal structure (see Section 3.1) and the complex tangent spaces to these $\mathbf{P}_3(\mathbb{C})$'s passing through a given point of Q realise the maximal isotropic subspaces of this conformal structure at that point.

Similarly, lying on Q_+ there are two families of $\mathbf{P}_3(\mathbb{C})$'s given by the images in $\mathbf{P}(\Sigma_+)$ of maximal isotropic subspaces of (Σ_+, B_+). These can be parametrised by the disjoint union $Q \cup Q_-$ in the following way: if $[v] \in Q$, then $\operatorname{Ker} v = \{\psi \in \Sigma_+ : v \cdot \psi = 0\}$ is maximal isotropic, and if $[\psi_-] \in Q_-$ then $V \cdot \psi_- \subset \Sigma_+$ is maximal isotropic. In analogous fashion the maximal isotropic subspaces of (Σ_-, B_-) are parametrised by $Q \cup Q_+$.

These observations can be refined (cf. [C]) to give:

Proposition 2.2.

(a) *Consider the two families Q_+ and Q_- of $\mathbf{P}_3(\mathbb{C})$'s on Q as above. Two $\mathbf{P}_3(\mathbb{C})$'s in the same family intersect in either a $\mathbf{P}_3(\mathbb{C})$, a $\mathbf{P}_1(\mathbb{C})$ or not at all. Two $\mathbf{P}_3(\mathbb{C})$'s in opposite families intersect in either a $\mathbf{P}_2(\mathbb{C})$ or a point.*

(b) *Let $A \subset V$ be an isotropic subspace of dimension r, where $1 \le r \le 4$ and let* $\operatorname{Ker}^{\pm} A = \operatorname{Ker} A \cap \Sigma_{\pm}$. *Then:*

  (i) $\dim \operatorname{Ker} A = 2^{4-r}$ *and* $\operatorname{Ker} A$ *is B_Σ-isotropic.*

  (ii) $\operatorname{Ker} A = \operatorname{Ker}^+ A \oplus \operatorname{Ker}^- A$ *and if $r < 4$*, $\dim \operatorname{Ker}^{\pm} A = 2^{4-r-1}$.

(c) *Let $\sigma : V \to V$ be a real form of (V, B) (i.e. σ is a conjugate linear involution such that $B(\sigma(v_1), \sigma(v_2)) = \overline{B(v_1, v_2)}$) and let $j : \Sigma \to \Sigma$ be a corresponding structure map (see above). If $[\psi] \in Q_+ \cup Q_-$ is a projective isotropic semi-spinor, then:*

  (i) $\operatorname{Ann}(j(\psi)) = \sigma(\operatorname{Ann}(\psi))$;

  (ii) $\operatorname{Ann}(\psi) \cap \sigma(\operatorname{Ann}(\psi))$ *is an isotropic subspace of V, stable under σ.*

Proof. For proofs of (a) and (b), see [C]. The statement (c)(i) follows immediately from the defining relation of j, namely $j(v \cdot \psi) = \sigma(v) \cdot j(\psi)$, and the statement (c)(ii) follows since the intersection of two isotropic subspaces is isotropic. □

2.1

Most of the information discussed above can be conveniently summarised in the following incidence diagrams:

$$
\begin{tikzcd}[column sep=small] & I_+ \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Q & & Q_+ \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & I_- \arrow[dl, "\pi_1"'] \arrow[dr, "p_1"] & \\ Q & & Q_- \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & I_- \arrow[dl, "\pi_2"'] \arrow[dr, "p_2"] & \\ Q_+ & & Q_- \end{tikzcd}
$$

For example, an element of I_+ is a couple $([x], \Pi)$, where $[x]$ is an isotropic line in V and Π is a $\mathbf{P}_3(\mathbb{C})$on Q in the family parametrised by Q_+ such that $[x] \in \Pi$. Alternatively we can think of elements of I_+ as couples $([x], [\psi])$, where $x \in V$, $\psi \in \Sigma_+$ satisfy $x \cdot \psi = 0$. The fibres of p are isomorphic to $\mathbf{P}_3(\mathbb{C})$ by definition and Proposition 2.2 shows that the fibres of π are also isomorphic to $\mathbf{P}_3(\mathbb{C})$.

2.2

If $x \in V$ is an isotropic vector, let $L_x \subset V$ be the line spanned by x and let $L_x^{\perp}$ be the orthogonal complement of L_x with respect to B. We will write K_x for $\operatorname{Ker} x = \{\psi \in \Sigma :$

$x \cdot \psi = 0\}$ and $K_x^{\pm}$ for $\operatorname{Ker} x \cap \Sigma_{\pm}$. Notice that the $K_x^{\pm}$ are B_{Σ}-isotropic of dimension four by Proposition 2.2(a).

If $y \in L_x^{\perp}$, then in the Clifford algebra of V we have $y \cdot x + x \cdot y = 0$ and so $y \cdot K_x \subseteq K_x$. Hence, since x acts trivially on K_x by Clifford multiplication, we have a map $f_x : L_x^{\perp}/L_x \to \operatorname{End}(K_x)$. The six-dimensional vector space $L_x^{\perp}/L_x$ inherits a symmetric, non-degenerate bilinear form from B, say B_x, since L_x is isotropic and it is clear that

$$f_x(\alpha) \circ f_x(\beta) + f_x(\beta) \circ f_x(\alpha) = -2B_x(\alpha, \beta)\,\mathrm{Id} \quad \forall \alpha, \beta \in L_x^{\perp}/L_x.$$

This means that the map f_x extends to an algebra homomorphism $\tilde{f}_x : C(L_x^{\perp}/L_x, B_x) \to \operatorname{End}(K_x)$, where $C(L_x^{\perp}/L_x, B_x)$ denotes the Clifford algebra of the inner product space $(L_x^{\perp}/L_x, B_x)$.

Proposition 2.3.
(i) *The map* $\tilde{f}_x : C(L_x^{\perp}/L_x, B_x) \to \operatorname{End}(K_x)$ *is an algebra isomorphism.*
(ii) *The decompositon* $K_x = K_x^+ \oplus K_x^-$ *is the decomposition into semi-spinor spaces.*

Proof. Exercise. □

Let (X, B) be a six-dimensional, non-degenerate, complex inner product space and let $S = S^+ \oplus S^-$ be an associated spinor space. Parts (a) and (b) of Proposition 2.4 describe the link between projective semi-spinors, maximal isotropic subspaces and isometric complex structures in this dimension. When (X, B) has a positive definite real form, part (c) describes how the space of isometric complex structures of a codimension two real subspace is embedded in (one component) of the space of isometric complex structures of X^{σ}.

Proposition 2.4.
(a) *The disjoint union of projective spaces* $\mathbf{P}(S^+) \cup \mathbf{P}(S^-)$ *naturally parametrises the space of maximal isotropic subspaces of* (X, B). *The parametrisation associates to* $[\psi] \in \mathbf{P}(S^+) \cup \mathbf{P}(S^-)$ *the maximal isotropic subspace* $\operatorname{Ann}(\psi) = \{w \in X : w \cdot \psi = 0\}$.
(b) *Let* σ *be a positive definite real form of* (X, B). *Then* $\operatorname{Ann}(\psi) \cap \sigma(\operatorname{Ann}(\psi)) = \{0\}$. *Furthermore, the map which associates to* $[\psi] \in \mathbf{P}(S^+) \cup \mathbf{P}(S^-)$ *the complex structure on* X^{σ} *whose space of* $(0,1)$*-vectors is* $\operatorname{Ann}(\psi) = \{w \in X : w \cdot \psi = 0\}$ *is a bijection of the disjoint union of projective spaces* $\mathbf{P}(S^+) \cup \mathbf{P}(S^-)$ *onto the space* $\mathfrak{J}(X^{\sigma}, B|_{X^{\sigma}})$ *of complex structures of the real vector space* X^{σ} *which are isometric for the restriction of* B *to* X^{σ}. *The orientation, say* ω_+, *of* X^{σ} *induced by a complex structure corresponding to an element of* $\mathbf{P}(S^+)$ *is opposite to the orientation of* X^{σ} *induced by a complex structure corresponding to an element of* $\mathbf{P}(S^-)$.
(c) *Let* σ *be a positive definite real form of* (X, B), *let* $W \subset X^{\sigma}$ *be a real four-dimensional subspace and let* $W^{\perp} \oplus \mathbb{C} = A \oplus \bar{A}$ *be the uniqe decomposition of its complex orthogonal complement as a sum of isotropic subspaces. Then:*
  (i) $\{[\psi] \in \mathbf{P}(S^+) : A \cdot \psi = 0$ *or* $\bar{A} \cdot \psi = 0\}$ *is the disjoint union of two linear* $\mathbf{P}_1(\mathbb{C})$*'s* $\{[\psi] \in \mathbf{P}(S^+) : A \cdot \psi = 0\}$ *and* $\{[\psi] \in \mathbf{P}(S^+) : \bar{A} \cdot \psi = 0\}$;

(ii) *the bijection of* (b) *maps* $\{[\psi] \in \mathbf{P}(S^+): A \cdot \psi = 0 \text{ or } \bar{A} \cdot \psi = 0\}$ *onto the space of isometric complex structures of* X^σ *which preserve the subspace* W *and which induce the orientation* ω_+;

(iii) *by restriction to* $W \subset X^\sigma$, *the space of isometric complex structures of* X^σ *which preserve the subspace* W *and which induce the orientation* ω_+ *is in bijection with the space* $\mathfrak{J}(W, B|_W)$ *of all isometric complex structures of* $(W, B|_W)$.

Remark 2.5. If D is a one-dimensional complex vector space and (X, B) is as in Proposition 2.4, then $X \otimes D$ has a natural conformal structure induced by the inner product of X. The maximal isotropic subspaces of X and $X \otimes D$ are then in natural bijection ($M \leftrightarrow M \otimes D$). If (X, B) and D have real structures, the space of complex structures of $(X \otimes D)^{\sigma'}$ preserving the conformal structure is in natural bijection with the space of isometric complex structures of X^σ, and thus also can be parametrised by $\mathbf{P}(S^+) \cup \mathbf{P}(S^-)$.

Proof of Proposition 2.4. More generally, it is true (see [C]) that for a non-degenerate inner product space of dimension four or six, the union of the spaces of projective semi-spinors parametrises the space of maximal isotropic subspaces in this way. This proves part (a).

Part (b) is also well known. The point is that for $\mathrm{Ann}(\psi)$ to be the space of $(0, 1)$-vectors of a complex structure on X^σ, it must satisfy $\mathrm{Ann}(\psi) \cap \sigma(\mathrm{Ann}(\psi)) = \{0\}$. This is true because otherwise there would exist non-zero real isotropic vectors, which is impossible when the signature is definite.

To prove part (c), left us denote by J_ψ the isometric complex structure on X^σ corresponding to $[\psi] \in \mathbf{P}(S^+)$ by (b). Then $A \cdot \psi = 0$ means that A is a space of $(0, 1)$-vectors for J_ψ so that $J|_A = -\mathrm{i}\,\mathrm{Id}_A$ and (by conjugation) $J|_{\bar{A}} = \mathrm{i}\,\mathrm{Id}_{\bar{A}}$. Hence J_ψ preserves the subspace $A \oplus \bar{A}$ of X and therefore also its orthogonal complement $W \otimes \mathbb{C}$. Since J_ψ is real it must also preserve W.

Conversely, given an isometric complex structure J of X^σ which induces the orientation ω_+ and preserves W, let $[\psi_J]$ be the corresponding element of $\mathbf{P}(S^+)$. Then J preserves the two-dimensional $W^\perp \otimes \mathbb{C}$ and so we have the decomposition as a direct sum of one-dimensional eigenspaces

$$W^\perp \otimes \mathbb{C} = \{u : J(u) = \mathrm{i}u\} \oplus \{u : J(u) = -\mathrm{i}u\},$$

and since J is isometric, the eigenspaces are isotropic. Comparing this with the decomposition

$$W^\perp \otimes \mathbb{C} = A \oplus \bar{A},$$

we must have $\{u : J(u) = -\mathrm{i}u\} = A$ or $\{u : J(u) = -\mathrm{i}u\} = \bar{A}$. Thus either A or $\bar{A}$ is of type $(0, 1)$ for J and thus either $A \cdot \psi_J = 0$ or $\bar{A} \cdot \psi_J = 0$.

The final part of (c) follows because $(W^\perp)^\sigma$, being two-dimensional, has exactly two isometric complex structures, say $J_{W^\perp}$ and $-J_{W^\perp}$. Hence any isometric complex structure J_W of $(W, B|_W)$ can be extended to an isometric complex structure $\widetilde{J_W}$ of X^σ inducing the orientation ω_+ by setting $\widetilde{J_W} = J_W \oplus J_{W^\perp}$ or $\widetilde{J_W} = J_W \oplus -J_{W^\perp}$, depending on which one of these induces the orientation ω_+. □

3. The conformal six-sphere as a real form of Q

3.1

Let us choose a real form σ of (V, B) of signature $(7, 1)$. This is by definition a conjugate linear involution $\sigma : V \to V$ such that $B(\sigma(v_1), \sigma(v_2)) = \overline{B(v_1, v_2)}\ \forall\, v_1, v_2 \in V$ and such that the restriction of B to the fixed point set of σ, V^σ, is of signature $(7, 1)$ (i.e. there are seven positive eigenvalues and one negative eigenvalue).

As explained in Section 2, σ induces a conjugate linear structure map on spinors $j : \Sigma \to \Sigma$, which in this signature satisfies $j^2 = \mathrm{Id}$ and $j(\Sigma_+) = \Sigma_-$, $j(\Sigma_-) = \Sigma_+$ (cf. [C]). This in turn induces an antiholomorphic involution of $Q_+ \cup Q_-$ sending Q_+ to Q_-, which we will also denote by j.

It is well known that the fixed point set of the induced action of σ on the projective isotropic cone Q is diffeomorphic to the six-sphere. Henceforth we will denote this sphere by S^6. It is also well known that the six-sphere realised in this way is naturally equipped with its standard conformal structure. This follows from the basic fact that there is a natural isomorphism of complex vector spaces

$$T_L Q \cong L^* \otimes_{\mathbb{C}} L^\perp / L, \tag{1}$$

where $L \in Q$ is an isotropic line and $T_L Q$ is the complex tangent space to the quadric Q at the point $L \in Q$. The vector space $L^\perp/L$ carries the induced non-degenerate inner product B_L (cf. Section 2.2) and this defines a non-degenerate inner product on $T_L Q$ up to a scalar factor via the isomorphism (1): in other words, the quadric Q has a natural complex conformal structure. If L is a real isotropic line in V, the real structure maps $L^\perp$ to $L^\perp$ and taking real points on both sides of (1) we get an isomorphism of real vector spaces at the point $L \in S^6$

$$T_L S^6 \cong (L^*)^\sigma \otimes_{\mathbb{R}} (L^\perp/L)^\sigma \tag{2}$$

and the complex conformal structure is reduced by σ to a real conformal structure of signature $(6, 0)$ on $T_L S^6$.

Conversely, it is natural to ask to what extent the objects V, Σ_+, Σ_-, Q, Q_+, Q_-, etc. are determined by the conformal geometry of S^6.

By Proposition 2.3, the vector bundle over Q whose fibre at $[x] \in Q$ is K_x and which we will denote by $K \to Q$, is a bundle of spinors for the vector bundle over Q whose fibre at $[x] \in Q$ is $(L_x^\perp/L_x, B_x)$ in the sense that we have natural isomorphisms $\tilde{f}_x : C((L_x^\perp/L_x), B_x) \cong \mathrm{End}(K_x)$ which depend algebraically on $[x]$. (By definition the bundles $\mathbf{P}(K^+) \to Q$ and $\mathbf{P}(K^-) \to Q$ are precisely the incidence bundles $\pi : I_+ \to Q$ and $\pi_1 : I_- \to Q$ of Section 2.1.)

The structure map $j : \Sigma \to \Sigma$ satisfies $j(K_x^+) = K_{\sigma(x)}^-$ and $j(K_x^-) = K_{\sigma(x)}^+$, and thus provides a antiholomorphic lift of the real structure $\sigma : Q \to Q$ to the bundle $K \to Q$. If $[x] \in S^6 \subset Q$ is real, the subspace $K_x \subset \Sigma$ is stable under $j : \Sigma \to \Sigma$ and so j induces a conjugate linear involution of the fibre K_x which exchanges K_x^+ and K_x^-.

By Proposition 2.4 and Remark 2.5, at each point $[x] \in S^6$ the disjoint union $\mathbf{P}(K_x^+) \cup \mathbf{P}(K_x^-)$ is isomorphic to the space of almost complex structures of $T_{[x]}S^6 \cong L^\sigma \otimes (L_x^\perp / L_x)^\sigma$ which preserve the conformal structure. Thus the disjoint union of projective bundles $\mathbf{P}(K^+) \cup \mathbf{P}(K^-) \to S^6$ is naturally isomorphic to the bundle of pointwise almost complex structures which are compatible with the pointwise conformal structure of S^6, and is therefore intrinsically associated to the conformal structure of S^6. Complex structures in the two different connected components $\mathbf{P}(K^+)$ and $\mathbf{P}(K^-)$ are distinguished by the fact that they induce opposite orientations on S^6.

Consider the bundle $\mathbf{P}(K^+) \to S^6$. For each $[x] \in S^6$ the fibre $\mathbf{P}(K_x^+)$ is a $\mathbf{P}_3(\mathbb{C})$ lying on the quadric Q_+ and we now show that as $[x]$ varies in S^6, this fibres the quadric Q_+ over the sphere S^6.

Proposition 3.1.

(a) *Suppose* $[x]$, $[y]$ *are distinct points of* S^6. *Then the maximal isotropic subspaces* K_x^+ *and* K_y^+ *of* Σ_+ *are disjoint:* $K_x^+ \cap K_y^+ = \{0\}$.

(b) *If* $[\psi] \in Q_+$, *then* $\{x \in V^\sigma : x \cdot \psi = 0\} = \mathrm{Ann}(\psi) \cap \sigma(\mathrm{Ann}(\psi))$ *is of dimension one.*

Proof.

(a) Suppose for contradiction that there exists a non-zero spinor $\psi \in K_x^+ \cap K_y^+$. Then by definition $x \cdot \psi = y \cdot \psi = 0$ and so $-2B(x, y)\psi = x \cdot y \cdot \psi + y \cdot x \cdot \psi = 0$. Thus $B(x, y) = 0$ and the vectors x and y are orthogonal, spanning a real, two-dimensional isotropic subspace. These do not exist in signature (7, 1) and we have a contradiction.

(b) Take $[\psi] \in Q_+$. By Proposition 2.2(c)(ii), $\mathrm{Ann}(\psi) \cap \sigma(\mathrm{Ann}(\psi))$ is an isotropic subspace of V which is stable under σ, i.e. it is the complexification of a real isotropic subspace of V^σ. Since we are in signature (7, 1), this real vector space is of dimension zero or one. On the other hand,

$$\sigma(\mathrm{Ann}(\psi)) = \mathrm{Ann}(j(\psi))$$

by Proposition 2.2(c)(i) and $j(\psi) \in \Sigma_-$ in signature (7, 1) as indicated above. Hence by Proposition 2.2(b), $\dim(\mathrm{Ann}(\psi) \cap \sigma(\mathrm{Ann}(\psi))) = \dim(\mathrm{Ann}(\psi) \cap (\mathrm{Ann}(j(\psi))))$ is equal to one or three and so in fact $\dim(\mathrm{Ann}(\psi) \cap \sigma(\mathrm{Ann}(\psi))) = 1$. □

The following proposition resumes the situation.

Proposition 3.2. *The map* $\tau : Q_+ \to S^6$ *given by*

$$\tau([\psi_+]) = [\{x \in V^\sigma : x \cdot \psi_+ = 0\}] = \mathrm{Ann}(\psi_+) \cap \sigma(\mathrm{Ann}(\psi_+))$$

fibres the quadric Q_+ *over* S^6 *with fibre* $\mathbf{P}_3(\mathbb{C})$ *and induces an orientation* ω_+ *on* S^6. *It is* C^∞*-isomorphic to one component of the bundle of pointwise almost complex structures of* S^6 *which are compatible with the conformal structure of* S^6 *and induce the orientation* ω_+.

Thus we know how to obtain the C^∞-manifold Q_+ from the conformal structure of S^6 plus an orientation, and the question now is how do we obtain its complex structure from this data. This is explained in the next section.

4. The twistor construction

In this section we will describe briefly the general twistor construction of which the fibration $\tau : Q_+ \to S^6$ of Section 3 is a special case. For more details and proofs see [AHS], [BeO], [I], [O'BR] or [S].

Definition 4.1. Let $(M, \omega, [g])$ represent a real $2n$-dimensional manifold M equipped with a positive-definite conformal structure $[g]$ and an orientation ω. The twistor space of $(M, \omega, [g])$ is the pair (Z^+, J^+) where:

(i) Z^+ is the total space of the fibre bundle over M whose fibre at $m \in M$ is the space of complex structures (henceforth abbreviated to CS) on T_mM which preserve the conformal structure $[g]$ and induce the orientation ω. Note that a fibre is isomorphic to the homogeneous manifold $\mathrm{SO}(2n)/U(n)$ and therefore has two (opposite) natural complex stuctures and a natural Kähler metric.

(ii) The pullback of the tangent bundle of M to Z^+ has a tautological CS and can be identified with the horizontal subspace of TZ^+ via the Levi-Cività connection of any metric in the conformal class. The almost complex structure (henceforth ACS) $J^+ : TZ^+ \to TZ^+$ is defined as the direct sum of this CS in horizontal directions and one of the natural CS of (i) in fibre directions (precisely which one is given in the references above).

Here are the basic properties of this construction.

Remark 4.2. The pair (Z^+, J^+) depends only on the conformal structure of M even though a metric was needed for the definition of J^+. Each fibre is canonically a Kähler manifold.

Remark 4.3. The ACS J^+ is integrable iff $(M, [g])$ is conformally flat (resp. half-conformally flat) when $\dim M > 4$ (resp. $\dim M = 4$). Then a fibre $Z^+(m)$ is a complex submanifold of Z^+ and the holomorphic normal bundle $N \to Z^+(m)$ satisfies $\dim H^0(N) = 2n$ and $H^1(N) = H^0(N^*) = H^1(N^*) = \{0\}$ (cf. [I] or [S]).

Remark 4.4. One can also define the 'anti-twistor' space (Z^-, J^-) of $(M, \omega, [g])$ as above but using CS which induce the orientation $-\omega$. The map $J \mapsto -J$ then defines an 'antiholomorphic' involution of (Z^+, J^+) if $\dim M = 4k$, and an antiholomorphic isomorphism of (Z^+, J^+) with (Z^-, J^-) if $\dim M = 4k + 2$. If $(M, [g])$ is a manifold with conformal structure (not oriented) then one can define the 'total' twistor space as above but using all pointwise CS which preserve the conformal structure. If M connected is not orientable this will be a connected manifold, but if M is orientable it will have two connected components Z^+ and Z^-.

Remark 4.5. The natural lift to Z^+ of an orientation preserving, conformal transformation of M preserves the ACS J^+.

Remark 4.6 (cf. [BdeB]). If $m \mapsto J(m)$ is a global ACS on M which preserves the conformal structure and induces the orientation ω, then it is integrable iff the tangent space of the submanifold $\{(m, J(m)) : m \in M\} \subset Z^+$ is stable under J^+.

By Section 3, the fibration $\tau : Q_+ \to S^6$ is smoothly isomorphic to the twistor fibration of (S^6, ω_+). It remains to be proved that the holomorphic structure of Q_+ corresponds to the holomorphic structure of twistor space described above. This is a straightforward verification based on the fact that the holomorphic structure on twistor space is the only one invariant by the action of the group of direct, conformal transformations of S^6. Using this we can give a conformal proof of the following theorem of LeBrun ([LeB]).

Theorem 4.7. *There does not exist an integrable, almost complex structure on S^6 which is compatible with the standard conformal structure.*

Proof. By the theorem of Burns–de Bartolomeis (i.e. Remark 4.6), such an almost complex structure would give rise to a complex submanifold of twistor space, evidently isomorphic to S^6. By the above discussion, the twistor space of S^6 is isomorphic to the quadric Q_+, which is projective and therefore Kähler. Hence any complex submanifold is also Kähler and therefore has non-trivial second cohomology group. This is not the case for S^6 so we have a contradiction. □

The following definition and simple proposition will be needed later.

Definition 4.8. Let $(M, [g], \omega)$ be as above, let $i : N \to M$ be a codimension two immersion of an oriented manifold N and let $T_{i(n)}M = i_*(T_nN) \oplus (i_*(T_nN))^{\perp}$ be the corresponding orthogonal decomposition of tangent spaces. Let $J^{\perp}$ and $-J^{\perp}$ be the two CS on $(i_*(T_nN))^{\perp}$ which preserve the induced conformal structure. If $Z^+(N) \to N$ is the twistor space of N with respect to the induced conformal structure and given orientation, the direct twistor lift of the map $i : N \to M$ is defined to be the map $\tilde{i}_+ : Z^+(N) \to Z^+(M)$ where

$$\tilde{i}(J_n) = J_n \oplus J^{\perp} \quad \text{or} \quad J_n \oplus -J^{\perp},$$

depending on whether $J_n \oplus J^{\perp}$ or $J_n \oplus -J^{\perp}$ induces the orientation ω. Similarly one defines the opposite twistor lift $\tilde{i}_- : Z^-(N) \to Z^+(M)$.

Proposition 4.9. *With $(M, [g], \omega)$ as above, let $F \in \Lambda^2(T^*_mM \otimes \mathbb{C})$ be a complex two form at $m \in M$ and let $\mathfrak{J}_+(m)$ denote the space of* CS *of T_mM which preserve the conformal structure and induce the orientation ω. For $J \in \mathfrak{J}_+(m)$ denote by $F_J^{0,2}$ the component of type $(0, 2)$ of F with respect to J. Then if* dim $M \geq 6$,

$$F_J^{0,2} = 0 \quad \forall J \in \mathfrak{J}_+(m) \quad \Leftrightarrow \quad F = 0.$$

Corollary 4.10. *Let $\pi : Z^+ \to M$ be the twistor fibration and suppose* $\dim M \geq 6$. *If $F \in C^\infty(\Lambda^2(T^*M))$ is a global two-form such that π^*F has no component of type* $(0, 2)$ *with respect to the almost complex structure J^+, then $F = 0$.*

Proof of Proposition 4.9. The subspace $\{F \in \Lambda^2(T_m^* \otimes \mathbb{C}) : F_J^{0,2} = 0\ \forall J \in \mathfrak{J}_+(m)\}$ of $\Lambda^2(T_m^*M \otimes \mathbb{C})$ is stable under the action of $\mathrm{SO}(T_mM, g_m)$, the group of direct orthogonal transformations of T_mM which preserve any metric in the conformal class. It is well known that the complex two forms are an irreducible representation space of the special orthogonal group if the dimension is greater than five and hence this set is either $\Lambda^2(T_m^*M \otimes \mathbb{C})$ or $\{0\}$, the first possibility being clear excluded. □

To prove Corollary 4.10 it is sufficient to note that if π^*F has no component of type $(0, 2)$, then at $m \in M$, $F(m) \in \Lambda^2(T_mM^* \otimes \mathbb{C})$ has no component of type $(0, 2)$ for all CS $J \in \mathfrak{J}_+(m)$ by the very definition of the almost complex structure J^+ on Z^+.

Remark 4.11. This proposition is not true in dimension four since there the space of two forms can be decomposed into the sum of self-dual and anti-self-dual forms (cf. [AHS]).

5. Linear $\mathbf{P}_3(\mathbb{C})$'s on Q_+ and the conformal geometry of S^6

Consider the twistor fibration $\tau : Q_+ \to S^6$. In Section 2 we saw that the quadric Q parametrises one of the families of linear $\mathbf{P}_3(\mathbb{C})$'s lying on Q_+. If $x \in V$ is a real isotropic vector, then $[x] \in Q$ is in the sphere S^6 and the corresponding $\mathbf{P}_3(\mathbb{C}) \subset Q_+$ is just the fibre $\tau^{-1}([x])$ as in Proposition 3.2. In this section we will consider the other members of the family Q.

Notation 5.1. Let $x \in V$ be a non-zero isotropic vector. We set $K_x^+ = \{\psi \in \Sigma_+ : x \cdot \psi = 0\}$ and $[K_x^+]$ will denote the image of K_x^+ in $\mathbf{P}(\Sigma)$. We denote by $\langle x, \sigma(x)\rangle$ the subspace of V spanned by x and $\sigma(x)$, by $\langle x, \sigma(x)\rangle^\perp$ its orthogonal complement with respect to B and by $[\langle x, \sigma(x)\rangle^\perp]$ the image of $\langle x, \sigma(x)\rangle^\perp$ in $\mathbf{P}(V)$.

Proposition 5.2.
(a) *If $x \in V$ is a non-zero isotropic vector, then*

$$\tau([K_x^+]) = [\langle x, \sigma(x)\rangle^\perp] \cap S^6;$$

(b) *if moreover x and $\sigma(x)$ are independent over $\mathbb{C}$, then the restriction of B to $(\langle x, \sigma(x)\rangle^\perp)^\sigma$ is of signature* $(5, 1)$.

Proof.
(a) Suppose $\psi \in K_x^+ - \{0\}$. Then $T([\psi]) = [\mathrm{Ann}\,\psi \cap \sigma(\mathrm{Ann}\,\psi)]$ by Proposition 3.2. The complex vector space $\mathrm{Ann}\,\psi \cap \sigma(\mathrm{Ann}\,\psi)$ is of dimension one and invariant under σ so we can choose a non-zero real vector $y \in V$ which generates it.

Since $y \cdot \psi = 0$ and $x \cdot \psi = 0$, we have $(x \cdot y + y \cdot x) \cdot \psi = 0$ and by the rules of Clifford multiplication this implies that $B(x, y) = 0$. Since y is a real vector, this in turn implies that $B(\sigma(x), y) = 0$. Hence $[y] \in [\langle x, \sigma(x)\rangle^{\perp}] \cap S^6$ and we have shown that $\tau([K_x^+]) \subseteq [\langle x, \sigma(x)\rangle^{\perp}] \cap S^6$.

To prove inclusion in the other direction, suppose that $[z]$ is an element of $[\langle x, \sigma(x)\rangle^{\perp}] \cap S^6$, where $z \in V$ is a non-zero real isotropic vector. Then x and z span either a one-dimensional or two-dimensional isotropic subspace of V and so by Proposition 2.2(c) there exists a non-zero isotropic spinor such that $x \cdot \psi = z \cdot \psi = \sigma(z) \cdot \psi = 0$. Thus there is a ψ in K_x^+ such that $z \in \mathrm{Ann}\,\psi \cap \sigma(\mathrm{Ann}\,\psi)$ and we are done.

(b) Suppose that $x \in V$ is an isotropic vector such that $\sigma(x) \neq \lambda x$ for any $\lambda \in \mathbb{C}$. Then $B(x, \sigma(x)) \neq 0$ because otherwise the subspace $\langle x, \sigma(x)\rangle \subset V$ would be the complexification of a two-dimensional isotropic subspace of V^{σ} and these do not exist in signature $(7, 1)$. This implies that the restriction of B to the real vector space $\langle x, \sigma(x)\rangle^{\sigma}$ is of signature $(2, 0)$ for an orthogonal basis is given by the vectors $x + \sigma(x)$ and $\mathrm{i}(x - \sigma(x))$. Hence the restriction of B to the real orthogonal complement $(\langle x, \sigma(x)\rangle^{\perp})^{\sigma}$ is of signature $(5, 1)$. □

Remark 5.3. From Proposition 5.2(a) it is immediate that

$$\tau([K_x^+]) = \tau([K_y^+]) \quad \Leftrightarrow \quad [x] = [y] \,\mathrm{or}\, [\sigma(x)] = [y].$$

Remark 5.4. Given a six-dimensional real subspace $W \subset V^{\sigma}$ such that $B|W$ is of signature $(5, 1)$, one can always find an isotropic vector $x \in V$ such that $\sigma(x) \neq \lambda x$ and $W = (\langle x, \sigma(x)\rangle^{\perp})^{\sigma}$. This line $[x]$ is determined up to conjugation by σ. To prove this observe that $(W^{\perp})^{\sigma}$ is a two-dimensional real vector space such that the restriction of B is of signature $(2, 0)$. For $[x]$ we take an isotropic line in $W^{\perp} \otimes \mathbb{C}$ and there are only two of these.

Proposition 5.2 gives a characterisation of the images under the twistor map $\tau : Q_+ \to S^6$ of the $\mathbf{P}_3(\mathbb{C})$'s on Q_+ parametrised by the quadric Q. There are two cases:

(i) if $\sigma(x) = \lambda x$—i.e. if $[x] \in S^6$—then $[\langle x, \sigma(x)\rangle] \cap S^6 = [x]$ and $\tau([K_x^+]) = [x]$. This says that the fibres of $\tau : Q_+ \to S^6$ are members of the family Q.

(ii) if $\sigma(x) \neq \lambda x$—i.e. if $[x] \in Q \backslash S^6$—then $\tau([K_x^+]) = [W] \cap S^6$, where W is a real six-dimensional subspace of V such that the restriction of B to W is of signature $(5, 1)$ Recalling that S^6 is the real projective isotropic cone of a real inner product space of signature $(7, 1)$, it is well known that $[W] \cap S^6$ is a conformal four-sphere in S^6 and that every conformal four-sphere in S^6 can be obtained by intersection with a unique such W.

It follows from (ii), Remarks 5.3 and 5.4 that the map $[x] \mapsto \tau([K_x^+])$ defines a $2:1$ covering of the space of conformal four-spheres in S^6 by the complex manifold $Q \backslash S^6$ or equivalently, an isomorphism of the complex manifold $Q \backslash S^6$ with the space of *oriented* conformal four-spheres in S^6. Now $Q \backslash S^6$ has a 'tautological' bundle with fibre type $\mathbf{P}_3(\mathbb{C})$whose fibre at $[x]$ is $[K_x^+]$ and the space of oriented conformal four-spheres in S^6 carries the tautological bundle with fibre S^4. The rest of this section is essentially devoted to showing that the map between these two tautological bundles induced by τ is fibre by fibre the Penrose twistor fibration.

In order to study case (ii), fix x in V such that $\sigma(x) \neq \lambda x$ and let us denote the conformal four-sphere $\tau([K_x^+])$ by S_x^4. We also denote by $\mathfrak{J}_+([x])$ the space of complex structures on $T_{[x]}S^6$ which preserve the conformal structure and induce the orientation ω_+, and by $\mathfrak{J}_{[x]}(W)$ the space of complex structures preserving the real subspace $W \subset T_{[x]}S^6$ and the induced conformal structure. Then we have:

Proposition 5.5.

(i) $K_x^+ \cap K_{\sigma(x)}^+ = K_x^- \cap K_{\sigma(x)}^- = \{0\}$.

(ii) *The restriction of the twistor map* $\tau : Q_+ \to S^6$ *to* $\tau : [K_x^+] \to S_x^4$ *is isomorphic to the Penrose twistor fibration for a suitable orientation of the conformal four-sphere* S_x^4. *Its restriction to* $\tau : [K_{\sigma(x)}^+] \to S_x^4$ *is then isomorphic to the Penrose twistor fibration for the opposite orientation of the conformal four-sphere* S_x^4.

(iii) *If* $[z] \in S_x^4$ *and if we identify* $\tau^{-1}([z])$ *with* $\mathfrak{J}_+([x])$, *then*

$$\begin{aligned}
&(\tau^{-1}([z]) \cap [K_x^+]) \cup (\tau^{-1}([z]) \cap [K_{\sigma(x)}^+]) \\
&\quad = \{J \in \mathfrak{J}_+([z]) : J(T_{[z]}S_x^4) = T_{[z]}S_x^4\} \\
&\quad \cong \mathfrak{J}_{[z]}(T_{[z]}S_x^4) \quad \textit{(by restriction)}.
\end{aligned}$$

Proof. Let us first remark that $B(x, \sigma(x)) \neq 0$ since otherwise $\langle x, \sigma(x)\rangle$ would be the complexification of a real, two-dimensional, isotropic subspace of (V, B) and these do not exist in signature $(7, 1)$.

To prove (i), suppose that ψ is a spinor such that $x \cdot \psi = \sigma(x) \cdot \psi = 0$. Then $x \cdot \sigma(x) \cdot \psi + \sigma(x) \cdot x \cdot \psi = 0$ and hence $-2B(x, \sigma(x))\psi = 0$ by the rules of Clifford multiplication. Since $B(x, \sigma(x)) \neq 0$, we have $\psi = 0$. This proves (i).

To prove (ii), let us realise $K_x^+ \oplus K_{\sigma(x)}^+$ as a space of spinors for the real Lorentz space (W_x^σ, B, σ) such that this decomposition is the decomposition into semi-spinor spaces. Here we write $W_x = \langle x, \sigma(x)\rangle^\perp$. Define the linear map $f : W_x \to C(V, B)$ by

$$f(z) = \frac{1}{\sqrt{2B(x, \sigma(x))}} z \cdot (x + \sigma(x)). \tag{3}$$

Then, taking the square of $f(z)$ in the complex Clifford algebra $C(V, B)$, we get

$$\begin{aligned}
f(z)^2 &= \frac{1}{2B(x, \sigma(x))} z \cdot (x + \sigma(x)) \cdot z \cdot (x + \sigma(x)) \\
&= \frac{1}{2B(x, \sigma(x))} (-z^2) \cdot (x \cdot \sigma(x) + \sigma(x) \cdot x) \\
&= \frac{1}{2B(x, \sigma(x))} (2B(z, z))(-2B(x, \sigma(x))) 1d \\
&= -2B(z, z) 1d,
\end{aligned}$$

using the relations $z \cdot x + x \cdot z = z \cdot \sigma(x) + \sigma(x) \cdot z = 0$ and $x^2 = \sigma(x)^2 = 0$. By the universal property of the Clifford algebra $C(W_x, B)$, the map f extends to an algebra homomorphism $\tilde{f} : C(W_x, B) \to C(V, B)$, which is injective since the algebra $C(W_x, B)$ is simple. It remains to show that for $z \in W_x$, $f(z)$ acting by Clifford multiplication in Σ sends K_x^+ to $K_{\sigma(x)}^+$ and vice versa.

If $\psi \in K_x^+$, that is if $x \cdot \psi = 0$, we have

$$\tilde{f}(z) \cdot \psi = z \cdot x \cdot \psi + z \cdot \sigma(x) \cdot \psi = z \cdot \sigma(x) \cdot \psi,$$

which is in $K_{\sigma(x)}^+$ since

$$\sigma(x) \cdot z \cdot \sigma(x) \cdot \psi = -z \cdot \sigma(x)^2 \cdot \psi = 0.$$

Similarly, $\tilde{f}(z) \cdot K_{\sigma(x)}^+ \subseteq K_x^+$. From this, we deduce that $\tilde{f}$ maps $C(W_x, B)$ into $\mathrm{End}(K_x^+ \oplus K_{\sigma(x)}^+)$ and this must in fact be an isomorphism for dimensional reasons. Thus, the space $K_x^+ \oplus K_{\sigma(x)}^+$ is a spinor space for the (5, 1) signature Lorentz space (W_x, B) and hence the map $\tau' : [K_x^+] \to S_x^4$ given by

$$\tau'([\psi]) = [\{w \in W_x^\sigma : \tilde{f}(w) \cdot \psi = 0\}]$$

is the Penrose twistor fibration for a suitable orientation of S_x^4 by Section 2. It remains to show that $\tau([\psi]) = \tau'([\psi])$ for $\psi \in K_x^+$, or in other words that

$$\{w \in W_x^\sigma : \tilde{f}(w) \cdot \psi = 0\} = \mathrm{Ann}(\psi) \cap (\sigma(\mathrm{Ann}(\psi))).$$

By formula (3) above, $\tilde{f}(w) \cdot \psi = 0$ means that $w \cdot \sigma(x) \cdot \psi = 0$ (since $x \cdot \psi = 0$), whence $\sigma(x) \cdot w \cdot \psi = 0$ because w and $\sigma(x)$ are orthogonal vectors. Similarly, $x \cdot w \cdot \psi = 0$. Thus $w \cdot \psi \in K_x^- \cap K_{\sigma(x)}^-$ and this is $\{0\}$ by part (i). Hence $w \in \mathrm{Ann}(\psi)$ and so $w \in \sigma(\mathrm{Ann}(\psi))$ since w is a real vector. This proves (ii).

Differentiating Proposition 5.2(a) identifies the subspace $T_{[z]}S_x^4 \otimes \mathbb{C}$ in $T_{[z]}S^6 \otimes \mathbb{C} \cong L_z^* \otimes L_z^\perp / L_z$ as

$$T_{[z]}S_x^4 \otimes \mathbb{C} \cong L_z^* \otimes \langle x, \sigma(x), z\rangle^\perp / L_z.$$

Let us define the one-dimensional isotropic subspaces $A, \bar{A}$ of $T_{[z]}S^6 \otimes \mathbb{C}$ by $A = L_z^* \oplus [x]$ and $\bar{A} = L_z^* \oplus [\sigma(x)]$, where $[x]$ is the equivalence class of the isotropic vector x in $L_z^\perp / L_z$. Then clearly $A \oplus \bar{A} = (T_{[z]}S_x^4 \otimes \mathbb{C})^\perp$ and we have the decomposition

$$T_{[z]}S^6 \otimes \mathbb{C} = T_{[z]}S_x^4 \oplus \mathbb{C} \oplus A \oplus \bar{A}.$$

Now, by Proposition 2.4(c), part (iii) is equivalent to proving that

$$\tau^{-1}([z]) \cap [K_x^+] = \{\psi : A \cdot \psi = 0\} \quad \text{and} \quad \tau^{-1}([z]) \cap [K_{\sigma(x)}^+] = \{\psi : \bar{A} \cdot \psi = 0\}.$$

But this follows from the definition of A and $\bar{A}$, and the fact that $x \cdot \psi = 0$ for $\psi \in K_x^+$ and $\sigma(x) \cdot \psi = 0$ for $\psi \in K_{\sigma(x)}^+$. □

Summarising the above results, we have proved the following theorem.

Theorem 5.6.

(i) *If P is a $P_3(\mathbb{C})$ belonging to the family parametrised by Q* (cf. Section 2), *then P is either a fibre of the fibration $\tau : Q_+ \to S^6$ or its image $\tau(P)$ is a conformal four* *sphere* (cf. [I]); *furthermore* $\tau : P \to \tau(P)$ *is isomorphic to the Penrose fibration for a suitable orientation of* $\tau(P)$.

(ii) *Over each conformal four sphere* S *in* S^6, *there are exactly two* $P_3(\mathbb{C})$*'s of the family* Q. *They induce opposite orientations on* S *and intersect each fibre of* $\tau : Q_+ \to S^6$ *over* S *in two disjoint* $P_1(\mathbb{C})$*'s. If we identify* $\tau : Q_+ \to S^6$ *with the twistor fibration, then these* $P_3(\mathbb{C})$*'s are the images of the twistor lifts* (*cf. Definition* 4.8) *of the inclusion* $S \hookrightarrow S^6$.

Remark 5.7. The theorem gives a description of one family of linear $P_3(\mathbb{C})$'s on the twistor space of S^6 in terms of the conformal geometry of S^6. If we take a $P_3(\mathbb{C})$ in the other family (parametrised by Q_-), it is easy to see that the map τ sends it onto S^6 and that the restriction of τ is injective except on a $P_2(\mathbb{C})$ which is collapsed to a point. Schematically, this is the map from $P_3(\mathbb{C}) = \mathbb{C}^3 \cup (P_2(\mathbb{C}))_\infty$ to $S^6 = \mathbb{R}^6 \cup \{\infty\}$ which maps $\mathbb{C}^3$ isomorphically onto $\mathbb{R}^6$ and the hyperplane at infinity to the point at infinity.

Remark 5.8. Any conformally flat, oriented six-manifold M is locally isomorphic to the standard six-sphere and the twistor construction produces a complex six-manifold $Z^+(M)$ which is fibred over M with fibre $\mathbf{P}_3(\mathbb{C})$, say $T_M : Z^+(M) \to M$. The image under T_M of a small deformation δ of the fibre Z^+_m at $m \in M$ is either a point or a conformal four-sphere S^4_δ in M and then the map $T_M : \delta \to S^4_\delta$ is the usual Penrose twistor fibration for a suitable orientation of S^4_δ. Conversely, given any oriented conformal four-sphere in M, its Penrose transform is a $\mathrm{P}_3(\mathbb{C})$ lying on $Z^+(M)$ by the twistor lift construction of Definition 4.8. Thus we can define an involution $\sigma_{M_\mathbb{C}}$ on the space $M_\mathbb{C}$ of small holomorphic deformations of fibres by

$$\sigma_{M_\mathbb{C}}(\delta) = \delta \quad \text{if } \delta \text{ is a fibre;}$$
$$\sigma_{M_\mathbb{C}}(\text{the twistor lift of } (S^4, \omega)) = \text{the twistor lift of } (S^4, -\omega).$$

This gives an intrinsic (that is in terms of the conformal geometry of M) definition of the real structure on $M_\mathbb{C}$ which was also defined in [I]. Hence we have an embedding of M in $M_\mathbb{C}$ as the fixed point set of $\sigma_{M_\mathbb{C}}$. Since the holomorphic normal bundle N to a fibre satisfies $\dim H^0(N) = 6$ and $H^1(N) = \{0\}$ (cf. Remark 4.3), $M_\mathbb{C}$ is a complex manifold by the theory of Kodaira and the tangent space at $\delta \in M_\mathbb{C}$ is naturally isomorphic to $H^0(\delta, N)$, the holomorphic sections of the normal bundle to δ in $Z^+(M)$. The sections which vanish somewhere define the isotropic cone of a complex conformal structure on $M_\mathbb{C}$. Locally the embedding $M \hookrightarrow (M_\mathbb{C}, \sigma_{M_\mathbb{C}})$ is isomorphic to the embedding $S^6 \hookrightarrow (Q, \sigma)$ of Section 3 and this implies that $\sigma_{M_\mathbb{C}}$ is antiholomorphic and that the complex conformal structure of $M_\mathbb{C}$ is reduced to a positive definite conformal structure over the fixed point set of $\sigma_{M_\mathbb{C}}$, that is over M.

This situation is similar to the case (cf. [AHS]) where M^4 is a self-dual four-manifold and one embeds M^4 in the space of $\mathbf{P}_1(\mathbb{C})$'s lying on its twistor space as the fixed point set of an antiholomorphic involution. However there is a difference: in the four-dimensional case, the antiholomorphic involution of the space of $\mathbf{P}_1(\mathbb{C})$'s on twistor space is induced by a natural holomorphic involution of twistor space whereas in the six-dimensional case there is no such natural involution on twistor space which induces σ_D on the space D.

6. Vector bundles on quadrics which are trivial on a linear $P_2(\mathbb{C})$

In this section we will prove the following theorem and deduce some corollaries.

Theorem 6.1. *If $\xi \to Q_+$ is a holomorphic vector bundle such that the restriction of ξ to the fibres of $\tau : Q_+ \to S^6$ is holomorphically trivial, then $\xi \to Q_+$ is holomorphically trivial.*

Proof. The Penrose/Ward transform (cf. [A,AHS]) sets up a correspondence between certain holomorphic bundles over $\mathbf{P}_3(\mathbb{C})$and bundles with self-dual connections over S^4. We will adapt the complex analytic version of this correspondence, of which the details are given in Hitchin's paper [H], to our case.

The main point of the proof is the following lemma.

Lemma 6.2. *The vector bundle $\mathcal{H}^0(\xi) \to S^6$ whose fibre at $p \in S^6$ is the complex vector space $H^0(\tau^{-1}(p), \xi)$ has a natural connection ∇.*

Proof. By hypothesis, dim $H^0(\tau^{-1}(p), \xi) = \text{rank } \xi$ is constant since ξ restricted to the fibres is holomorphically trivial. Hence the assignation $p \mapsto H^0(\tau^{-1}(p), \xi)$ does indeed define a smooth vector bundle over S^6 by elliptic regularity. To ease the notation, we will write F_p for $\tau^{-1}(p)$ from now on.

As explained in [A,H], a section of the bundle ξ over the first-order formal neighbourhood, denoted $F_p^{(1)}$, of F_p in Q_+ gives rise to an element of $J_p^1(\mathcal{H}^0(\xi))$, the one-jet bundle of $\mathcal{H}^0(\xi)$ at p. A connection in any vector bundle $\eta \to B$ can be seen as a splitting of the exact sequence

$$0 \to \Lambda^1 \otimes \eta \to J^1(\eta) \to \eta \to 0,$$

so to define a connection in the vector bundle $\mathcal{H}^0(\xi)$ it is sufficient to show that each section of ξ over F_p can be uniquely extended to a section of ξ over the first-order formal neighbourhood of F_p in Q_+. For more details of this argument, see Section 3 in Ch. VI of [A] or [H].

Now the appropriate exact sequence of sheaves over Q_+ for this extension problem (cf. [A,H]) is

$$0 \to \mathcal{O}_{F_p}(N^* \otimes \xi) \to \mathcal{O}_{F_p}^{(1)}(\xi) \to \mathcal{O}_{F_p}(\xi) \to 0,$$

where N^* is the holomorphic conformal bundle of F_p in Q_+, $\mathcal{O}_{F_p}^1(\xi)$ is the sheaf of (germs of) sections of ξ over $F_p^{(1)}$ and $\mathcal{O}_{F_p}(\xi)$ is the sheaf of (germs of) sections of ξ over F_p. The map $\mathcal{O}_{F_p}^{(1)}(\xi) \to \mathcal{O}_{F_p}(\xi)$ is just restriction. Taking cohomology we get

$$0 \to H^0(\mathcal{O}_{F_p}(N^* \otimes \xi)) \to H^0(\mathcal{O}^{(1)}_{F_p}(\xi))$$
$$\to H^0(\mathcal{O}_{F_p}(\xi)) \to H^1(\mathcal{O}_{F_p}(N^* \otimes \xi)) \cdots$$

By hypothesis, $\xi | F_p$ is holomorphically trivial and so $H^0(\mathcal{O}_{F_p}(N^* \otimes \xi)) = \{0\}$ iff $H^0(\mathcal{O}_{F_p}(N^*)) = \{0\}$ and $H^1(\mathcal{O}_{F_p}(N^* \otimes \xi)) = \{0\}$ iff $H^1(\mathcal{O}_{F_p}(N^*)) = \{0\}$. By Remark 4.3, for the normal bundle $N \to F_p$ we have $H^0(N^*) = H^1(N^*) = \{0\}$. Substituting these results in the cohomology sequence shows that the restriction map $H^0(\mathcal{O}^{(1)}_{F_p}(\xi)) \to H^0(\mathcal{O}_{F\,p}(\xi))$ is an isomorphism and the lemma is proved.

We now have a connection ∇ in $\mathcal{H}^0(\xi) \to S^6$ which we will show is flat. The first point is that the pullback bundle $\tau^*(\mathcal{H}^0(\xi)) \to Q_+$ is naturally identified with $\xi \to Q_+$ by evaluation. Thus $\xi \to Q_+$ carries the connection $\tau^*(\nabla)$. The second point is that $\tau^*(\nabla)$ is compatible with the holomorphic structure of $\xi \to Q_+$. This follows mutatis mutandis from the Remark after Corollary 3.8 in [H]. Hence the curvature from $F(\tau^*(\nabla))$ has no component of type (0, 2)—this is precisely the compatiblity condition. But $F(\tau^*(\nabla)) = \tau^*(F(\nabla))$ is the pullback of a two-form on S^6 and so by Corollary 4.10, $F(\nabla) = F(\tau^*(\nabla)) = 0$.

The holomorphic vector bundle $\xi \to Q_+$ therefore has a flat connection which is compatible with its holomorphic structure. Since Q_+ is simply connected, we can find a global trivialisation of covariantly constant sections and these are holomorphic by compatibility. Hence $\xi_+ \to Q_+$ is holomorphically trivial. □

Corollary 6.3. *If $\xi \to Q_+$ is a holomorphic vector bundle such that the restriction of ξ to a linear $\mathbf{P}_2(\mathbb{C})$ is holomorphically trivial, then $\xi \to Q_+$ is holomorphically trivial.*

Proof. Recall from Section 2 that the quadrics Q and Q_- parametrise two families of linear $\mathbf{P}_3(\mathbb{C})$'s on Q_+ and that the family Q contains the fibres of the map $\tau : Q_+ \to S^6$. In the course of the proof we will refer to linear $\mathbf{P}_1(\mathbb{C})$'s and linear $\mathbf{P}_2(\mathbb{C})$'s lying on $Q_+ (\subset \mathbf{P}_7(\mathbb{C}))$ as lines and planes respectively.

The idea of the proof is to show that the hypothesis of the corollary implies that ξ restricted to each fibre of $\tau : Q_+ \to S^6$ is trivial and then apply the theorem. We will need the following result of Barth [B].

(A) If $\xi \to \mathbf{P}_n(\mathbb{C})$ is a holomorphic vector bundle whose restriction to some $\mathbf{P}_2(\mathbb{C})$ is holomorphically trivial, then $\xi \to \mathbf{P}_n(\mathbb{C})$ is holomorphically trivial;
and the classical facts (cf. Proposition 2.2)

(B) If P_1 and P_2 are two linear $\mathbf{P}_3(\mathbb{C})$'s on Q_+ in the same family, then either $P_1 = P_2$, $P_1 \cap P_2 = 0$ or $P_1 \cap P_2$ is a line.

(C) If P_1 and P_2 are two linear $\mathbf{P}_3(\mathbb{C})$'s on Q_+ in different families, then $P_1 \cap P_2$ is either a point or a plane.

(D) If P is a linear $\mathbf{P}_3(\mathbb{C})$ on Q_+ and Π is a plane contained in P, then there is a unique P' in the other family of $\mathbf{P}_3(\mathbb{C})$'s such that $P \cap P' = \Pi$.

Now suppose that Π_1 is a plane on Q_+ such that $\xi | \Pi_1$ is trivial. Choose a $\mathbf{P}_3(\mathbb{C})$in the family parametrised by Q_-, say P_-, which contains Π_1. By (A), $\xi | P_-$ is trivial and by Remark 5.7, P_- intersects one fibre of $\tau : Q_+ \to S^6$ in a plane and the others in exactly one point. Clearly ξ is trivial on the exceptional fibre by (A). Let F be a fibre which intersects P_-

in a point, say $P_- \cap F = \{x\}$ and let Π_2 be any plane such that $x \in \Pi_2 \subset P_-$. By (D), there is a unique $\mathbf{P}_3(\mathbb{C})$, say P_+, in the family parametrised by Q such that $P_+ \cap P_- = \Pi_2$ and then $\xi | P_+$ is trivial by (A). But $P_+ \cap F$ contains a point, namely x, and so in fact contains a line, say L, by (B). Let Π_3 be any plane in F which contains this line L and let P'_- be the unique (using (D)) $\mathbf{P}_3(\mathbb{C})$in the family parametrised by Q_- such that $F \cap P'_- = \Pi_3$. By (C), we see that $P_+ \cap P'_-$ is a plane since it contains the line L. We already know that $\xi | P_+$ is trivial so that $\xi | P'_-$ is trivial by (A). Thus $\xi | \Pi_3$ is trivial and since Π_3 is a plane in F, it follows from (A) again that $\xi | F$ is trivial.

In conclusion, we have shown that ξ is trivial on every fibre so applying the theorem, the corollary follows. □

By induction this result can be extended to give:

Theorem 6.4. *Let G be a non-degenerate, complex quadric hypersurface of dimension greater than or equal to six. If $\xi \to G$ is a holomorphic vector bundle such that ξ restricted to a linear $\mathbf{P}_2(\mathbb{C})$ is holomorphically trivial, then $\xi \to G$ is holomorphically trivial.*

Proof. Corollary 6.3 is the result for a six-dimensional quadric. Suppose that it also holds for the n-dimensional quadric ($n \geq 6$) and let us deduce that it then holds for the $n+1$-dimensional quadric.

To fix notation, let (V, B) be an $n+3$-dimensional complex vector space V, equipped with a non-degenerate, symmetric bilinear form B. Then our model for the $n+1$-dimensional quadric will be

$$G_{n+1} = \{L \in \mathbf{P}(V) \colon B|L = 0\}.$$

The intersection of this quadric with a hyperplane is a quadric of dimension n, which may be degenerate. If H_v denotes the hyperplane $\{x \in V : B(x, v) = 0\}$, then $H_v \cap G_{n+1}$ is a non-degenerate quadric of dimension n if and only if $B(v, v) \neq 0$. Let G_n be any non-degenerate quadric of dimension n lying on G_{n+1} which contains the plane P on which ξ is trivial. (Such a G exists when $n \geq 4$). The induction hypothesis implies that $\xi | G_n$ is trivial.

By definition the quadric G_n is the zero locus of a section of the standard positive line bundle $\mathcal{O}(1)$ (over G_{n+1}) so the ideal sheaf of G_n is isomorphic to $\mathcal{O}_{G_{n+1}}(-1)$. Hence we have the exact sequence of sheaves on G_{n+1} (corresponding to the restriction of functions on G_{n+1} to functions on G_n)

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{G_n} \to 0,$$

where $\mathcal{O}$ denotes the structure sheaf $\mathcal{O}_{G_{n+1}}$ for brevity. Tensoring with the locally free sheaf of germs of sections of holomorphic sections of ξ, we get the exact sequence

$$0 \to \mathcal{O}(\xi(-1)) \to \mathcal{O}(\xi) \to \mathcal{O}_{G_n}(\xi) \to 0, \qquad (R)$$

and taking cohomology gives the long exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}(\xi(-1))) \to H^0(\mathcal{O}(\xi)) \to H^0(\mathcal{O}_{G_n}(\xi)) \\ &\to H^1(\mathcal{O}(\xi(-1))) \to H^1(\mathcal{O}(\xi)) \to H^1(\mathcal{O}_{G_n}(\xi)) \cdots \end{aligned}$$

Lemma 6.5. *Every holomorphic section of ξ over G_n extends uniquely to a holomorphic section of ξ over G_{n+1}; that is, the map $H^0(\mathcal{O}(\xi)) \to H^0(\mathcal{O}_{G_n}(\xi))$ in the above sequence is an isomorphism.*

Proof. Tensoring the sequence (R) by $\mathcal{O}(-k)$, where $k \geq 1$, gives

$$0 \to \mathcal{O}(\xi(-k-1)) \to \mathcal{O}(\xi(-k)) \to \mathcal{O}_{G_n}(\xi(-k)) \to 0,$$

and taking cohomology gives the long exact sequence

$$\begin{aligned} 0 \to\ & H^0(\mathcal{O}(\xi(-k-1))) \to H^0(\mathcal{O}(\xi(-k))) \to H^0(\mathcal{O}_{G_n}(\xi(-k))) \\ \to\ & H^1(\mathcal{O}(\xi(-k-1))) \to H^1(\mathcal{O}(\xi(-k))) \to H^1(\mathcal{O}_{G_n}(\xi(-k))) \cdots \end{aligned} \qquad (R_k)$$

For $k \geq 1$, $H^0(\mathcal{O}_{G_n}((-k)))$ vanishes because $\mathcal{O}_{G_n}((-k))$ is a negative line bundle and the group $H^1(\mathcal{O}_{G_n}((-k))) \cong H^{n-1}(\Omega^n(k))^*$ is also zero by Serre duality and the Kodaira–Nakano vanishing theorem (cf. [GH]). Since ξ restricted to $G_n$is trivial, the same groups vanish for ξ restricted to G_n and so we have $H^0(\mathcal{O}_{G_n}(\xi(-k))) = H^1(\mathcal{O}_{G_n}(\xi(-k))) = \{0\}$ when $k \geq 1$. Substituting this in the cohomology sequence we get

$$H^1(\mathcal{O}(\xi(-k-1))) \cong H^1(\mathcal{O}(\xi(-k))) \quad \text{for } k \geq 1,$$

or in other terms

$$H^1(\mathcal{O}(\xi(-1))) \cong H^1(\mathcal{O}(\xi(-2))) \cong \cdots \cong H^1(\mathcal{O}(\xi(-k))) \cong \cdots$$

By another theorem of Kodaira (Theorem B in [GH]), $H^1(\mathcal{O}(\xi(-k))) = \{0\}$ for k large enough and so all of these groups vanish. In particular the first one is zero and this is exactly the condition for the map $H^0(\mathcal{O}(\xi)) \to H^0(\mathcal{O}_{G_n}(\xi))$ of (R) to be surjective. The lemma is proved.

To see that the restriction map is injective, substituting $H^1(\mathcal{O}_{G_n}(\xi(-k))) = \{0\}$ in (R_k) we get

$$H^0(\mathcal{O}(\xi(-k-1))) \cong H^0(\mathcal{O}(\xi(-k))) \quad \text{for } k \geq 1.$$

By a Kodaira vanishing theorem, exactly as above, this implies that $H^0(\mathcal{O}(\xi(-k))) = \{0\}$ for $k \geq 1$, and in particular that $H^0(\mathcal{O}(\xi(-1))) = \{0\}$. Substituting this in the cohomology sequence (R_1) shows that the restriction map $H^0(\mathcal{O}(\xi)) \to H^0(\mathcal{O}_{G_n}(\xi))$ is an isomorphism. □

From Lemma 6.5 it follows that there are exactly r global holomorphic sections of $\xi \to G_{n+1}$ where r is the rank of ξ. To prove that $\xi \to G_{n+1}$ is holomorphically trivial we have to show that at each point of G_{n+1} the values of these sections form a basis of the fibre at that point. This is equivalent to proving that a holomorphic section which vanishes somewhere vanishes everywhere. We need the following lemma.

Lemma 6.6. *Let P be a plane lying on the quadric G_{n+1} and let x be a point of G_{n+1}. Then there exists a non-degenerate n-dimensional quadric G' in G_{n+1} which contains both the plane P and the point x.*

Proof. Suppose first that x is not in P. The plane P is the projectivisation of a three-dimensional, isotropic linear subspace of V, say W. If L is the isotropic line of V generated by x, then $L \cap W = \{0\}$ by hypothesis and the orthogonal complement of $L \oplus W$ is of dimension $n - 1$. When $n \geq 6$, this is too large to be an isotropic subspace of V (which are of dimension at most $\frac{1}{2}(n + 3)$). Hence there exists a non-isotropic vector $y \in V$ which is orthogonal to both L and W and then the associated hyperplane $H_y = \{v \in V : B(y, v) = 0\}$ intersects G_{n+1} in a non-degenerate n-dimensional quadric, which contains both the plane P and the point x. When x is in P, a similar argument works. □

Proof of Theorem 6.4 (continued). Suppose s is a global holomorphic section of $\xi \to G_{n+1}$ which vanishes at some point x. If $P \subset G_{n+1}$ is the plane on which ξ is supposed trivial, there exists a non-degenerate, n-dimensional quadric G' containing x and P by Lemma 6.6. The induction hypothesis implies that ξ is trivial on G' and therefore the section s must vanish along G' since it vanishes at $x \in G'$. In particular s vanishes along P. Now if y is any point of G_{n+1}, we can find a non-degenerate quadric of dimension n, say G'', containing the point y and the plane P, again using Lemma 6.6. The induction hypothesis implies that ξ is trivial on G'' and therefore the section s must vanish along G'' since it vanishes at $P \subset G'$. In particular s vanishes at y and therefore s vanishes everywhere since y was arbitrary. □

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